Exactly solvable models of statistical mechanics and quantum field theory

Authors:

S.E. Derkachov[1], G.A. Sarkissian[2], and V.P. Spiridonov[2]

[1] – St. Petersburg Department of Steklov Mathematical Institute of Russian Academy of Sciences
[2] – BLTP JINR

The presented series of papers consists of 21 publications.

Confinement describes a transition from the quarks and gluons, which interact asymptotically free in the high-energy region, to the composite objects - baryons and mesons, which strongly interact in the low-energy region. The central problem of the theory of strong interactions consists in the explanation of this phenomenon. In 2008, Dolan and Osborn have shown that the sector of the so-called BPS-states in the simplest four-dimensional non-abelian N=1 supersymmetric field theory coincides with the sector of such states in the Wess-Zumino type model describing meson fields without gauge interaction. In this way they gave a partial justification of the confinement. This statement follows from the equality of superconformal indices of the corresponding theories tied to each other by the Seiberg electromagnetic duality, which is described by the exact evaluation formula for the elliptic beta integral mathematically rigorously proved by V.P. Spiridonov in 2000.

The Seiberg duality lies in the heart of the mechanism responsible for the exact solvability of the models of statistical mechanics generalizing the two-dimensional Ising model and one-dimensional Heisenberg spin chain, which are the basic models of the theory of phase transitions and the theory of magnetism. This statement was justified in the 2010 paper by V.P. Spiridonov, which is the first paper in the presented series. Namely, the equality of superconformal indices of dual theories identically coincides with the mathematical star-triangle relation either in the operator form (through the elliptic Fourier transformation and corresponding Bailey lemma, proven by V.P. Spiridonov) or in the functional form (through the equality of partition functions of elementary cells in the form of a “star” and a “triangle”', which was used by V.V. Bazhanov and S.M. Sergeev in the investigation of the most general model of the Ising type). Thus, the famous star-triangle relation appears to be connected with the confinement phenomenon in the theory of strong interactions.

The presented series of papers contains deep investigation and further development of the structural elements of this fundamental connection between the exactly solvable models of statistical mechanics and quantum field theory. At the same time it contains a development of various aspects of the general formalism of quantum integrable systems, which was developed earlier by S.E. Derkachov, and the conformal field theory which was investigated in the papers by G.A. Sarkissian. Main results obtained in the presented series of papers is listed below.

A fundamental connection between the Seiberg duality for N=1 supersymmetric quantum field theories in four-dimensional space-time with two-dimensional exactly solvable models of statistical mechanics is discovered. The star-triangle relation coincides with the evaluation of univariate elliptic beta integral and describes the equality of superconformal indices of the simplest dual theories, which proves the confinement in the sector of BPS states of the theory with the gauge group SU(2) and flavor group SU(6). Superconformal indices of quiver field theories describe partition functions of two-dimensional models of the Ising type with the spins in the vertices taking continuous values.

It is shown that the elliptic Fourier transformation operator defines an intertwining operator of equivalent representations of the Sklyanin and elliptic modular double algebras. The Bailey lemma for it proves the

star-triangle relation in the operator form. The most general rank 1 known solution of the Yang-Baxter equation is found in the form of the integral operator with an elliptic hypergeometric kernel.

A new wide class of finite-dimensional representations of the Sklyanin and elliptic modular double algebras is found. It leads to new integrable lattice systems generalizing Baxter's 8-vertex model. A degeneration of the elliptic modular double to the hyperbolic level is found which generalizes the Faddeev modular double.

A star-triangle relation is constructed which describes the equality of partition functions for dual supersymmetric three-dimensional theories, which generalizes the star-triangle relation of the Faddeev-Volkov model, as well as an analogous relation based on the superconformal indices of dual three-dimensional theories.

A new integrable many-body model of the Ruijsenaars type is constructed which is obtained by a new degeneration of the van Diejen many-body model. A new generalization of the fundamental Selberg integral is constructed, which is formulated in terms of the complex beta integrals in the Mellin-Barnes representation.

It was shown that 6j-symbols of the Faddeev's modular double and SL(2,C) group are obtained from the elliptic hypergeometric function (the superconformal index of a simplest non-abelian gauge theory which is dual to an analogous theory) by sequential limiting transitions. New degeneration are found of the hyperbolic integrals (partition functions of three-dimensional supersymmetric theories) to complex and rational hypergeometric functions corresponding to the two-dimensional conformal field theory with the central charge values $c = 1$ and $c = 25$.

Parafermionic (rarefied) generalizations of the elliptic hypergeometric integrals, i.e. of the superconformal indices of theories on the four-dimensional space-time, related to the special lens space are constructed. A similar generalization was found for the rank 1 solution of the Yang-Baxter equation in the form of an integral operator. The exact formula for evaluation of the univariate hyperbolic beta integral related to the general lens space was established as well.

A parafermionic generalization of the hyperboilic function describing the most important part of the fusion matrix for the Liouville field theory and Racah-Wigner symbols for the Faddeev modular double is constructed. It emerges in a certain limit from the rarefied elliptic hypergeometric function. Its symmetries and mixed recurrence-difference equations are established. The simplest case corresponds to the supersymmetric hypergeometric function connected with N=1 supersymmetric Liouville theory and Racah-Wigner symbolds for the quantum algebra $U_q(osp(1|2))$.

List of papers:

1. G. A. Sarkissian and V. P. Spiridonov, Elliptic and complex hypergeometric integrals in quantum field theory, Phys. Part. Nucl. Lett. 20, no. 3 (2023), 281-286.
2. E. Apresyan, G. Sarkissian and V. P. Spiridonov, A parafermionic hypergeometric function and supersymmetric 6j-symbols, Nucl. Phys. B 990 (2023), 116170, 28 pp.
3. S. E. Derkachov, G. A. Sarkissian and V. P. Spiridonov, Elliptic hypergeometric function and 6j-symbols for the SL(2,C) group, Teor. Mat. Fiz. 213:1 (2022), 108-128. (С. Э. Деркачев, Г. А. Саркисян, В. П. Спиридонов «Эллиптическая гипергеометрическая функция и 6j-символы для группы SL(2,C)», ТМФ 213:1 (2022), 108-128.)
4. G. A. Sarkissian and V. P. Spiridonov, Complex hypergeometric functions and integrable many-body problems, J. Phys. A: Math. Theor. 55 (2022), 385203.

5. G. Sarkissian and V. Spiridonov, Elliptic, hyperbolic, complex gamma functions and QFT in various dimensions, Proc. of the RDP school and workshop “Aspects of Symmetry”, POS (Regio2021) (2022), 037.

6. G. A. Sarkissian and V. P. Spiridonov, Rational hypergeometric identities, Functional Analysis and Its Appl. 55:3 (2021), 91-97. (Г. А. Саркисян, В. П. Спиридонов «Рациональные гипергеометрические тождества», Функц. Анализ и его Прил. 55:3 (2021), 91-97.)

7. G. A. Sarkissian and V. P. Spiridonov, General modular quantum dilogarithm and beta integrals, Proc. Steklov Math. Inst. 309 (2020), 251-270. (Г. А. Саркисян, В. П. Спиридонов «Общий модулярный квантовый дилогарифм и бета-интегралы», Труды МИАН 309 (2020), 269--289.)

8. G. A. Sarkissian and V. P. Spiridonov, The endless beta integrals, SIGMA 16 (2020), 074, 21 pp.

9. G. A. Sarkissian and V. P. Spiridonov, Modular group and the hyperbolic beta integral, Russian Math. Surveys 75 (2020), 575-577. (Г. А. Саркисян, В. П. Спиридонов «Модулярная группа и гиперболический бета-интеграл», УМН 75:3 (2020), 187-188.)

10. V. P. Spiridonov, The rarefied elliptic Bailey lemma and the Yang-Baxter equation, J. Phys. A: Math. and Theor. 52 (2019), 355201.

11. S. E. Derkachov and V. P. Spiridonov, The 6j-symbols for the SL(2,C) group, Theor. Math. Phys. 198 (1) (2019), 29-47. (С. Э. Деркачев, В. П. Спиридонов «О $6j$-символах для группы SL(2,C)», ТМФ 198:1 (2019), 32-53.)

12. G. Sarkissian and V. P. Spiridonov, From rarefied elliptic beta integral to parafermionic star-triangle relation, J. High Energy Physics 10 (2018), 097.

13. V. P. Spiridonov, Rarefied elliptic hypergeometric functions, Advances in Mathematics 331 (2018), 830-873.

14. K. Yu. Magadov and V. P. Spiridonov, Matrix Bailey lemma and the star-triangle relation, SIGMA 14 (2018), 121.

15. D. Chicherin and V. P. Spiridonov, The hyperbolic modular double and Yang-Baxter equation, Advanced Studies in Pure Mathematics 76 (2018), 95-123; arXiv:1511.00131

16. D. Chicherin, S. E. Derkachov, and V. P. Spiridonov, New elliptic solutions of the Yang-Baxter equation, Commun. Math. Phys. 345 (2016), 507-543.

17. D. Chicherin, S. E. Derkachov, V. P. Spiridonov, From principal series to finite-dimensional solutions of the Yang-Baxter equation, SIGMA 12 (2016), 028, 34 pp.

18. S. E. Derkachov and V. P. Spiridonov, Finite dimensional representations of the elliptic modular double, Theor. Math. Phys. 183 (2) (2015), 597-618. (С. Э. Деркачев, В. П. Спиридонов «Конечномерные представления эллиптического модулярного дубля», ТМФ 183:2 (2015), 177-201.)

19. Gahramanov and V. P. Spiridonov, The star-triangle relation and 3d superconformal indices, J. High Energy Physics 08 (2015), 040.

20. S. E. Derkachov and V. P. Spiridonov, Yang-Baxter equation, parameter permutations, and the elliptic beta integral, Russian Math. Surveys, 68:6 (2013), 1027-1072. (С. Э. Деркачев, В. П. Спиридонов «Уравнение Янга-Бакстера, перестановки параметров, и эллиптический бета-интеграл», УМН 68:6 (2013), 59-106.)

21. V. P. Spiridonov, Elliptic beta integrals and solvable models of statistical mechanics, in: Algebraic Aspects of Darboux Transformations, Quantum Integrable Systems and Supersymmetric Quantum Mechanics, Contemp. Math. 563 (2012), 181--211; arXiv:1011.3798